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Discrete Fourier-series method in problems of bending of variable-thickness rectangular plates

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Abstract. The approach to the solution of the boundary-value problems of bending of elastic rectangular plates of variable thickness is presented. It is proposed to introduce into the resolving system of partial differential equations additional functions which enables the variables to be formally separated and the problem to be reduced to a unidimensional one by representing all the functions as a Fourier series in a single coordinate. In this case the problem can be solved by the stable numerical method of discrete orthogonalization. To calculate the additional functions, Fourier series of discretely assigned functions with allowance for variations in the plate thickness are used. The boundary-value problems for rectangular plates of variable thickness were solved assuming that their weight is unchanged.

Key words: discrete orthogonalization method, discretely specified functions, Fourier series, rectangular plates, variable thickness

1. Introduction

Plates of variable thickness along with plates of uniform thickness, see Timoshenko [1, Chapters 5–6] , are widely used as structural elements in different fields of engineering. It seems to be interesting to evaluate their stress-strain behavior depending on thickness variation. However, with allowance for the thickness variation in one or two coordinate directions, the mathematical model and solution of the corresponding boundary-value problem become complicated. In this case the boundary-value problem of bending of the variable-thickness rectangular plate is described, as was demonstrated by Grigorenko and Mukoed [2, pp. 76–81], by a system of partial differential equations with variable coefficients and certain boundary conditions. When the thickness is uniform, the problem is reduced to a biharmonic equation for deflection. A large list of literature on solving the problem of bending of rectangular plates of uniform thickness is given in a review by Meleshko, Gomilko and Gourjii [3].

One of the approaches to solving problems of the given class for a plate of uniform thickness under certain boundary conditions according to Timoshenko [1, Chapters 5–6], Kantorovich and Krylov [4, Chapter 4.2.4] consists in representing the sought-for solution as a generalized Fourier series along one of coordinates and in reducing the initial problem to a unidimensional one for amplitude values. If the plate thickness varies along one coordinate direction, then the solution may be represented as a series in a coordinate along the other coordinate direction, and upon separation of variables, the problem can be reduced to a unidimensional one. In the case where the thickness varies along two coordinate directions, the boundary-value problem can not be reduced to a unidimensional one.

This paper considers an approach to the solution of a two-dimensional boundary-value problem by means of the reduction of the initial problem to a unidimensional one by expanding the solution into a Fourier series in one coordinate and determining some unknown functions that enter into the obtained system of differential equations, representing these as a Fourier series in discretely assigned values; see Grigorenko and Timonin [5]. The unidimensional boundary-value problem is integrated by the discrete orthogonalization numerical method proposed by Godunov [6], Grigorenko [7, pp. 80–84, 90–94], Bellman and Kalaba [8, Chapter 4.21].

2. Statement of the problem

We will consider the two-dimensional boundary-value problem involving the bending of a thin elastic rectangular plate of variable thickness along two coordinate directions using the linear theory of thin plates that is based on the hypothesis of the undeformed Kirchhoff normal [1, Chapter 4]. The middle surface of the plate is associated with the Cartesian rectangular coordinate system XOY. Let us assume that the plate is under the action of a uniform load which is applied normally to the middle surface. Using expressions for the curvatures that are written in terms of a normal displacement, equilibrium equations and elasticity relations, we represent the governing system of partial differential equations in the form [2, pp. 76–81]

$$\begin{aligned}
&\frac{\partial \widehat{Q}_y}{\partial y} = \frac{\partial^2}{\partial x^2}\left[D_M\left(1-\nu^2\right)\frac{\partial^2 w}{\partial x^2} - \nu M_y\right] - q\,,\\
&\frac{\partial M_y}{\partial y} = \widehat{Q}_y - 2\frac{\partial}{\partial x}\left[D_M\,(1-\nu)\,\frac{\partial \vartheta_y}{\partial x}\right],\\
&\frac{\partial w}{\partial y} = -\vartheta_y\,, \qquad \frac{\partial \vartheta_y}{\partial y} = \frac{1}{D_M}M_y + \nu\frac{\partial^2 w}{\partial x^2}\,,
\end{aligned} \tag{1}$$

where x and y are rectangular coordinates; the plate occupies a domain ($0 \le x \le a$; $0 \le y \le b$), $D_M = Eh^3\,(x, y)\,/[12\left(1-\nu^2\right)]$ is the flexural stiffness, $\widehat{Q}_y = Q_y + \partial H/\partial x$ is the reduced lateral force, M_y, H, w and ϑ_y are the bending and twisting moments, displacement and angle of rotation of the normal at $y = \text{const}$, respectively, $h = h\,(x, y)$ is the plate thickness, $q = q\,(x, y)$ is the load, E is Young's modulus and ν is Poisson's ratio.

System (1) can by reduced to a single equation with respect to w

$$\begin{aligned}
&D_M\triangle\triangle w + 2\frac{\partial D_M}{\partial x}\frac{\partial}{\partial x}\triangle w + 2\frac{\partial D_M}{\partial y}\frac{\partial}{\partial y}\triangle w + \triangle D_M\triangle w -\\
&-(1-\nu)\left(\frac{\partial^2 D_M}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D_M}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D_M}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = q\\
&(0 \le x \le a; 0 \le y \le b)\,,
\end{aligned} \tag{2}$$

that in turn can be considered as a generalized biharmonic equation. When the thickness h of the plate is constant, Equation (2) takes the form

$$\triangle\triangle w = q/D_M\,, \tag{3}$$

where $\triangle$ is the Laplace operator.

It is known [1, Chapter 5], [2, pp. 76–81] that in the case when all the plate edges are simply supported, for example, at $x = \text{const}$, the problem solution can be found in the form of a double Fourier series (the Navier method) by representing the load q and deflection w as a double Fourier series. If the plate thickness does not depend on one of the coordinates, for example, on x, that is $h = h(y)$, then the unknown functions of the system of Equations (1) and the load q can be represented as a univariable Fourier series along the coordinate x. This makes it possible to reduce the problem to one in a single dimension for the amplitude values of the series [1, Chapters 5], [2, pp. 76–81] (the Levi method).

In the problem under consideration, we assume that the boundary conditions at edges of the plate have the form

$$w = 0,\, M_x \equiv \nu M_y - \frac{Eh^3}{12}\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0 \quad \textit{and } a, \tag{4}$$

and the surface load is assigned to be

$$q = \sum_{n=1}^{N_1} q_n\,(y) \sin \frac{\pi n}{a} x\,. \tag{5}$$

The boundary conditions at $y = 0$ and b are arbitrary.

3. Method for solving the problem

In the case considered the conditions of simple support at the opposite edges of the plate when $x = 0$ and a are assigned. However, the variation in the plate thickness along the coordinate OX does not make it possible to separate the variables in x. In order to overcome these difficulties, let us use the approach based on the introduction of the additional functions into the governing system of equations. The values of the functions are defined by their expansion into the discrete Fourier series [9, Chapter 4.4.11], [10, Chapter 6], [11, Chapter 19.7].

In doing so, the original system of Equations (1) can be written as

$$\begin{gathered}
\frac{\partial \widehat{Q}_y}{\partial y} = \frac{E}{12}\frac{\partial^2 \varphi_1}{\partial x^2} - \nu \frac{\partial^2 M_y}{\partial x^2} - q\,, \\
\frac{\partial M_y}{\partial y} = \widehat{Q}_y - \frac{E}{6\,(1+\nu)}\frac{\partial \varphi_2}{\partial x}\,, \\
\frac{\partial w}{\partial y} = -\vartheta_y\,, \qquad \frac{\partial \vartheta_y}{\partial y} = \frac{12\left(1-\nu^2\right)}{E}\varphi_3 + \nu \frac{\partial^2 w}{\partial x^2}\,,
\end{gathered} \tag{6}$$

where three additional functions

$$\varphi_1\,(x, y) = h^3 \frac{\partial^2 w}{\partial x^2}, \quad \varphi_2\,(x, y) = h^3 \frac{\partial \vartheta_y}{\partial x}, \quad \varphi_3\,(x, y) = \frac{M_y}{h^3} \tag{7}$$

have been inserted.

The solution of the boundary-value problem for the system of Equations (6) with the boundary conditions (4) and the load (5) is searched for as expansions

$$\Phi\,(x,y)=\sum_{n=1}^{N}\Phi_n\,(y)\sin\lambda_n x,\ \lambda_n=\frac{\pi n}{a}\ , \tag{8}$$

$$\varphi_2\,(x,y)=\sum_{n=1}^{N}\varphi_{2,n}\,(y)\cos\lambda_n x\ ,\ \Phi=\left\{\widehat{Q}_y, M_y, w, \vartheta_y, \varphi_1, \varphi_3\right\},\ (N\geq N_1)\ .$$

Having substituted the expansions (8) in Equations (6), the boundary conditions (4) and the functions φ_i $(i=1,2,3)$ (7), we will obtain the following system of ordinary differential equations:

$$\frac{d\widehat{Q}_{y,n}}{dy}=-\lambda_n^2\left(\frac{E}{12}\varphi_{1,n}-\nu M_{y,n}\right)-q_n\ ,\qquad \frac{dM_{y,n}}{dy}=\widehat{Q}_{y,n}+\frac{E}{6(1+\nu)}\lambda_n\varphi_{2,n}\ ,$$
$$\frac{dw_n}{dy}=-\vartheta_{y,n},\quad \left(n=\overline{1,N}\right)\ ,\qquad \frac{d\vartheta_{y,n}}{dy}=\frac{12\left(1-\nu^2\right)}{E}\varphi_{3,n}-\nu\lambda_n^2 w_n\ , \tag{9}$$

with respect to amplitude values of these expansions.

The boundary conditions (4) at the edges $x=0$ and a are satisfied automatically. The boundary condition at the edges $y=0$ and b can also be expressed in terms of amplitude values of the resolving functions. For instance, if those edges are clamped, we have

$$w_n=0,\quad \vartheta_{y,n}=0\quad \text{at}\quad y=0\quad \text{and}\quad b,\ (n=\overline{1,N})\ . \tag{10}$$

The case of simply supported edges leads to

$$w_n=0,\quad M_{y,n}=0\quad \text{at}\quad y=0\quad \text{and}\quad b,\ (n=\overline{1,N})\ . \tag{11}$$

With allowance for expressions (7), for the amplitude values of additional functions entering into the solving equation system we obtain the relations

$$\varphi_{1,n}=\varphi_{1,n}\,(y,w_l)\ ,\quad \varphi_{2,n}=\varphi_{2,n}\left(y,\vartheta_{y,l}\right)\ , \tag{12}$$
$$\varphi_{3,n}=\varphi_{3,n}\left(y,M_{y,l}\right),\quad (l=\overline{1,N})\ .$$

These relations determine the connectedness of all $4N$ equations of the system (9).

As can be seen, the additional functions enter into the system of Equations (9) along with the solution functions, that is, the number of unknown functions exceeds the number of equations. Because of this, the functions (12) are to be calculated by integrating system (9). To do this, one can use the expansion of the noted functions into a discrete Fourier series in the coordinate x.

To find the values of the functions $\varphi_{j,n}\,(x,y)$ $\left(j=1,2,3;n=\overline{1,N}\right)$ in the course of the integration of the system (9), the current values of the amplitude of the resolving functions were used. To this end for each fixed point $y=y_k$ $(k=\overline{0,K})$ of the segment $[0,b]$ the magnitudes

$$h_i=h\,(x_i,y_k)\ ,$$

$$\varphi_1^i=\varphi_1\,(x_i,y_k)=-h_i^3\sum_{n=1}^{N}\lambda_n^2 w_n\,(y_k)\sin\lambda_n x_i\ ,$$

$$\varphi_2^i = \varphi_2\left(x_i, y_k\right) = h_i^3 \sum_{n=1}^{N} \lambda_n \vartheta_{y,n}\left(y_k\right) \cos \lambda_n x_i \,, \tag{13}$$

$$\varphi_3^i = \varphi_3\left(x_i, y_k\right) = \frac{1}{h_i^3} \sum_{n=1}^{N} M_{y,n}\left(y_k\right) \sin \lambda_n x_i \,,$$

were calculated at the set of points $x_i (i = \overline{1, S})$ of the segment $[0, a]$.

As a result, we have determined the values of the functions $\varphi_j(x_i, y_k)$ $(j = 1, 2, 3)$ calculated at points x_i $(i = \overline{1, S})$. With the obtained values we can expand these functions into a discrete Fourier-series representation in a form similar to (8). As this takes place, the coefficients of the series are defined as the missing amplitude values of additional functions for the system (9).

So, we extend the functions φ_1^i and φ_3^i, that are specified at the points x_i $\left(i = \overline{1, S}\right)$ of the segment $[0, a]$ through the odd range, and the function φ_2^i specified by the even range we extend to the segment $[a, 2a]$. Then, we calculate the values $\varphi_{j,n}$ $(j = 1, 2, 3; n = \overline{1, N})$ using to this end the standard procedure for determining the Fourier coefficients of the function [9, Chapter 4.4.11], [10, Chapter 6], [11, Chapter 19.7] that is specified in the table form. The obtained values of $\varphi_{j,n}(y_k)$ are substituted in the initial equation system (9) and integration is continued in y passing from the point y_k to the point $y_{k+1.}$

3.1. Discrete orthogonalization method

To solve the boundary-value problem for the system of Equations (9) with the boundary conditions (10), (11) we use the stable numerical method of discrete orthogonalization [6], [7, pp. 80–82, 90–94], [8, Chapter 4.21] based on the solution of the set of Cauchy problems by the Runge-Kutta method [9, Chapter 20.20.7] with their orthogonalization at the number of points along the axis OY. The essence of the method consists in the following.

Let us examine the linear boundary-value problem

$$\frac{\mathrm{d}\overline{Q}}{\mathrm{d}y} = A\left(y\right) \overline{Q}\left(y\right) + \overline{f}\left(y\right), \quad \left(0 \le y \le b\right), \tag{14}$$

with the boundary conditions

$$B_1 \overline{Q}\left(0\right) = \overline{b}_1 \,, \tag{15}$$

$$B_2 \overline{Q}\left(b\right) = \overline{b}_2 \,, \tag{16}$$

where $\overline{Q} = [\widehat{Q}_{y,1}, M_{y,1}, w_1, \vartheta_{y,1}; ...; \widehat{Q}_{y,N}, M_{y,N}, w_N, \vartheta_{y,N}]^T$ is the column vector, $A(y)$ is the square matrix of dimension $4N$ which has square blocks of dimension 4 along the diagonal; B_1 and B_2 are the rectangular matrices of dimension $P \times N$ and $(N - P) \times N$, respectively, $\overline{b}_1$ and $\overline{b}_2$ are the vectors of the right-hand side; $\overline{f} = [-\lambda^2{}_1 E/12\varphi_{1,1} - q_1, E/[6(1+\nu)]\lambda_1\varphi_{2,1}, 0, 12(1-\nu^2)/E\varphi_{3,1}; ...; -\lambda^2{}_{N_1} E/12\varphi_{1,N_1} - q_{N_1}, E/[6(1+\nu)]\lambda_{N_1}\varphi_{2,N_1}, 0, 12(1-\nu^2)/E\varphi_{3,N_1}; ...; -\lambda^2{}_N E/12\varphi_{1,N} - q_N, E/[6(1+\nu)]\lambda_N\varphi_{2,N}, 0, 12(1-\nu^2)/E\varphi_{3,N}]^T$ is the column vector of the right side.

It would be possible to solve the boundary-value problem (14)–(16) by a simple reduction to the set of Cauchy problems which are solved by some numerical method, for example, by the Runge-Kutta method [9, Chapter 20.20.7]. In this connection it should be noted that during the numerical solution of the boundary-value problems of the plate-and-shell theory, boundary and local effects are arising which cause a rapid growth of the searched functions. Because of this, the method of reduction of the boundary-value problems to Cauchy problems leads to numerical instability.

From a mathematical point of view it means that the real parts of matrix eigenvalues differ significantly from each other. Under some conditions this can cause loss of accuracy or even give rise to a completely incorrect solution. To overcome these difficulties, a number of methods have been developed by which the numerical solution of the boundary-value problems is reduced to a stable computational process. One such method, that has showed itself to be of advantage in solving the boundary-value problems, is the method of discrete orthogonalization [6], [7, pp. 80–82, 90–94], [8, Chapter 4.21].

In solving the boundary-value problem (14)–(16) we will search for a solution of the form

$$\overline{Q}\,(y) = \sum_{j=1}^{m} C_j \overline{Q}_j\,(y) + \overline{Q}_{m+1}\,(y)\ . \tag{17}$$

Here $\overline{Q_j}$ are the solution vectors of the Cauchy problems of the system of Equations (14) at $\overline{f(y)} = 0$ that satisfy the conditions (15) at $\overline{b}_1 = 0$, and $\overline{Q_{m+1}}$ is the solution vector of the Cauchy problem of the system (14) at $\overline{f(y)} \neq 0$ which satisfies the conditions (15) at $\overline{b}_1 \neq 0$. In this case $m = N - P$ is the number of boundary conditions at the right end of the integration interval $y = b$.

Among the integration points $y_k (k = 0, 1, ..., K)$ we choose the orthogonalization points $T_i (i = 0, 1, ..., M)$. The choice of the mentioned points usually is stipulated by the degree of the required accuracy of the problem solution, otherwise it is unrestricted.

Let at the point T_i the solutions of the Cauchy problem be obtained by any suitable numerical method, for example, the Runge-Kutta method. These solutions are represented in terms of $\overline{u}_r\,(T_i)\,(r = 1, 2, ..., m + 1)$.

Thus, at the point T_i before orthogonalization we have the vectors

$$\overline{u}_1\,(T_i)\,, \overline{u}_2\,(T_i)\,, ..., \overline{u}_m\,(T_i)\,, \overline{u}_{m+1}\,(T_i)\ . \tag{18}$$

Let us orthonormalize the vectors $\overline{u}_j\left(T_j\right)\ \ (j = 1, 2, ..., m)$ at the point T_i and designate these as

$$\overline{z}_1\,(T_i)\,, \overline{z}_2\,(T_i)\,, ..., \overline{z}_m\,(T_i)\ . \tag{19}$$

The vectors $\overline{z}_i$ can be expressed in terms of the vectors $\overline{u}_i$ as follows

$$z_r = \frac{1}{\omega_{rr}} \left[\overline{u}_r - \sum_{j=1}^{r-1} \omega_{rj} \overline{z}_j \right] \quad (r = 1, 2, ..., m)\ , \tag{20}$$

where

$$\omega_{rj} = \left(\overline{u}_r, \overline{z}_j\right) \quad (j < r)\,, \quad \omega_{rr} = \sqrt{(\overline{u}_r, \overline{u}_r) - \sum_{j=1}^{r-1} \omega_{rj}^2}\ .$$

The vector $\overline{z}_{m+1}$ can not be subjected to normalization and is calculated by the formula

$$\overline{z}_{m+1} = \overline{u}_{m+1} - \sum_{j=1}^{m} \omega_{m+1,j}\overline{z}_{j.} \,. \tag{21}$$

Based on the Equations (20) and (21) at $y = T_i$, we obtain

$$\begin{bmatrix} \overline{u}_1 (T_i) \\ \overline{u}_2 (T_i) \\ . \\ . \\ . \\ \overline{u}_m (T_i) \\ \overline{u}_{m+1} (T_i) \end{bmatrix} = \Omega_i \begin{bmatrix} \overline{z}_1 (T_i) \\ \overline{z}_2 (T_i) \\ . \\ . \\ . \\ \overline{z}_m (T_i) \\ \overline{z}_{m+1} (T_i) \end{bmatrix}, \tag{22}$$

where

$$\Omega_i = \Omega(T_i) = \begin{bmatrix} \omega_{11}(T_i) & 0 & 0 & \ldots & 0 \\ \omega_{21}(T_i) & \omega_{22}(T_i) & 0 & \ldots & 0 \\ \omega_{31}(T_i) & \omega_{32}(T_i) & \omega_{33}(T_i) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ \omega_{m1}(T_i) & \omega_{m2}(T_i) & \omega_{m3}(T_i) & \ldots & 0 \\ \omega_{m+1,1}(T_i) & \omega_{m+1,2}(T_i) & \omega_{m+1,3}(T_i) & \ldots & 1 \end{bmatrix}. \tag{23}$$

The vectors $\overline{z}_r (T_i)$ are the initial values of the Cauchy problems for the homogeneous $(r = 1, 2, ..., m)$ and inhomogeneous $(r = m + 1)$ systems of differential Equations (14) in the interval $T_i \leq y \leq T_{i+1}$.

At every orthogonalization point T_i the solution of the system of Equations (14) that satisfies the boundary conditions at the left-hand end of the interval (15) can be written as two expressions: before orthogonalization

$$\overline{Q}(T_i) = \sum_{j=1}^{m} C_j^{(i-1)}\overline{u}_j (T_i) + \overline{u}_{m+1} (T_i) \tag{24}$$

and after orthogonalization

$$\overline{Q}(T_i) = \sum_{j=1}^{m} C_j^{(i)}\overline{z}_j (T_i) + \overline{z}_{m+1} (T_i) \,. \tag{25}$$

The solution of the system of Equations (14) at the interval
$T_i \leq y \leq T_{i+1}$ can be represented in the form

$$\overline{Q}(y) = \sum_{j=1}^{m} C_j^{(i)}\overline{z}_j (y) + \overline{z}_{m+1} (y) \,. \tag{26}$$

After integration at the last section $T_{M-1} \leq y \leq T_M$ and orthogonalization at the point T_M by the formula (25) we have

$$\overline{Q}(T_M) = \sum_{j=1}^{m} C_j^{(M)}\overline{z}_j (T_M) + \overline{z}_{m+1} (T_M) \,. \tag{27}$$

Satisfying the boundary conditions at the right end of the integration interval, that is substituting Equation (27) in Equation (16), we obtain a system of m linear algebraic equations for determining the unknown values $C_j^{(M)}$ $(j = 1, 2, ..., m)$. After the values of $C_j^{(M)}$ are found, the solution of the boundary-value problem (14)–(16) at the point $y = T_M$ is given in the form (27). This completes forward running of the problem solution.

In the inverse solution of the problem the values of the constants $C_j^{(i-1)}$ $(j = 1, 2, ..., m)$ are determined by constants $C_j^{(i)}$ beginning from $i = M$. For this purpose we set equal the right sides of Equations (24) and (25):

$$\sum_{j=1}^{m} C_j^{(i-1)} \overline{u}_j (T_i) + \overline{u}_{m+1} (T_i) = \sum_{j=1}^{m} C_j^{(i-1)} \overline{z}_j (T_i) + \overline{z}_{m+1} (T_i) \ .d \tag{28}$$

Substituting values $\overline{u}_j$ from (22) and putting equal the coefficients for vectors $\overline{z}_j$ $(j = 1, 2, ..., m+1)$, we find

$$\Omega_i' \overline{C}^{(i-1)} = \overline{C}^{(i)} \quad (i = 1, 2, ..., M) \quad \text{or} \quad \overline{C}^{(i-1)} = \left[\Omega_i'\right]^{-1} \overline{C}^{(i)}, \tag{29}$$

where Ω_i' is the transposed matrix (23); $\overline{C}^{(i)}$ is the column vector with components $c_1^{(i)}, c_{2,...,}^{(i)} c_m^{(i)}, 1$.

In such a manner, by applying the equalities (29), the values of constants $C_j^{(i)}$ at all the points, beginning with $i = M$, are found. The solutions $\overline{Q}(T_i)$ for the boundary-value problem are calculated by the formula (26).

It should be noted that a computer realization of the given algorithm makes it necessary to store the information about the matrices Ω_i and vectors $\overline{z}_r$ $(r = 1, 2, ..., m+1)$.

In the practice of solving the problems the information obtained at all orthogonalization points usually is not used. In this case it is restricted only by the values of the searched functions at the set of points that are known as output points. The number of these points is considerably less than the number of orthogonalization points. In this connection we can essentially reduce the volume of the stored information, using the following procedure.

Let T_{i-1} and T_{i+p} be the output points. From the equality (29) one can obtain

$$\Omega_{i+p}' \Omega_{i+p-1}' ... \Omega_i' \overline{C}^{(i-1)} = \overline{C}^{(i+p)}, \tag{30}$$

hence

$$\overline{C}^{(i-1)} = \left[\left(\prod_{j=0}^{p} \Omega_{i+j} \right)' \right]^{-1} \overline{C}^{(i+p)}. \tag{31}$$

So, to determine the vector $\overline{C}^{(i-1)}$, it is necessary to store the information on the product of the matrices $\prod_{j=0}^{p} \Omega_{i+j}$. This gives a substantial saving of computer memory.

3.2. Numerical solution of the problem

At the start of the integration at $y = y_0 = 0$, when boundary conditions (10) or (11) are considered, the initial values of solution functions for the $2N + 1$ Cauchy problems are given.

Using the initial values for each of the $2N + 1$ Cauchy problems, we calculate the values of the functions $\varphi_{j,n}\left(j = 1, 2, 3; n = \overline{1, N}\right)$ by the formula (13) at $y = 0$. When the above-mentioned procedure is carried out, we find the amplitude values of the resolving functions for the coordinate $y = y_1$, which in turn are used for determining the amplitude values of the missing additional functions $\varphi_{j,n}(j = 1, 2, 3)$ that are needed to carry out the next integration step. By this means we integrate in y using the step $\Delta y_k = y_{k+1} - y_k(k = \overline{0, K-1})$ and orthogonalize at specified points of the interval $[0, b]$.

4. Numerical results and discussion

Let us consider the problem of the bending of a rectangular plate bending in order to describe the proposed algorithm and to estimate the accuracy of the solution obtained. The middle surface of the plate occupies the domain $0 \leq x \leq a, \quad 0 \leq y \leq b$. The variation in the plate thickness is described by the law

$$h(x) = h_0[1 + \alpha(1 - 6x + 6x^2)] \quad (-1 \leq \alpha \leq 1)\,. \tag{32}$$

The plate is subjected to the normal load

$$q(x, y) = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}\,. \tag{33}$$

All edges of the plate are simply supported; thus the boundary conditions can be written as (4),(11).

To solve the problem two approaches are used. The first one (1) is based on the proposed technique. The second one (2) is based on separation of variables in y, using Fourier series and the solution of the one-dimensional boundary-value problem is obtained by the stable discrete orthogonalization technique.

In carrying out the given approach the next steps should be taken. At the first step the set of points $x_i(i = \overline{1, S})$ is specified to calculate the values of the functions $\varphi_j \quad (j = 1, 2, 3)$ in the segment $0 \leq x \leq a$. The second step consists in the odd extension of the functions φ_1, φ_3 and in the even extension of the function φ_2 to the segment $a \leq x \leq 2a$. As a result we obtain the value of the functions $\varphi_j \quad (j = 1, 2, 3)$ at $R = 2(S - 1)$ points. Using these values, using a standard procedure, one can find the expansion of discrete given functions into Fourier series, as well as the coefficients of the expansion. In so doing, depending on a number of points R with allowance for some first coefficients of the expansion, we obtain the series sum with sufficient accuracy. For example, it is well known [11, Chapter 19.7] that for $R = 24$ an expansion can be limited by the first eight terms. More terms of the Fourier-series expansion can be included as the number of points R increases.

The method of discrete orthogonalization was used at the second step in the solution of the one-dimensional boundary-value problem. The number of orthogonalization points was chosen such that the required accuracy could be reached. The accuracy of the solution obtained was estimated by comparison of the results for an increasing number of orthogonalization points.

The problem was solved for $a = b = 1$; $h_0 = 0{\cdot}1$; $\alpha = 0{\cdot}3$. Dependence of the deflection w, angle of rotation ϑ_x and bending moment M_x on a number of points R and the number of expansion terms taken into account are listed in Table 1.

Table 1. Evaluation of solution convergence

V	R	N	wE/q_0		$-\vartheta_x E/q_0$		$M_x 10^2/q_0$	
			$x = 0{\cdot}1$	$x = 0{\cdot}5$	$x = 0{\cdot}0$	$x = 0{\cdot}4$	$x = 0{\cdot}1$	$x = 0{\cdot}5$
1	40	8	8·330	31·401	83·148	38·665	0·359	2·816
	60	15	8·345	31·434	83·251	38·666	0·378	2·816
	80	15	8·350	31·446	83·308	38·674	0·377	2·816
	100	15	8·352	31·452	83·334	38·677	0·378	2·817
2	–	–	8·357	31·463	83·377	38·687	0·379	2·817

Due to the symmetry of w and M_x and antisymmetry of ϑ_x, these values are listed only for some magnitudes of the x-coordinate. The values of the functions calculated within the framework of the second approach are shown in the last row. These values can be treated as exact. As this takes place, the number M of orthogonalization points is equal to 21 and 41. Using the table data as the base, one can estimate the solution convergence when R and N are increased. For instance, an agreement with exact solution to four significant figures is observed for $R = 100$ and $N = 15$.

Then, let us consider the problem of the bending of a rectangular plate in the domain $0 \leq x \leq a$ with the thickness varying along two coordinate directions by the law

$$h(x, y) = h_0[1 + \alpha(1 - 6x + 6x^2)][1 + \beta(1 - 6y + 6y^2)] \tag{34}$$

$$(-1 \leq \alpha \leq 1, \quad -1 \leq \beta \leq 1)\,.$$

From (34) it follows that for all values of the coefficients α and β the requirements on keeping the plate weight unchanged are satisfied. Similar problems have been solved in authors' paper [12].

The plate is acted upon by the normal load (33), the edges $x = 0$ and a are simply supported, while the edges $y = 0$ and b are clamped. As this takes place, the boundary conditions (4) and (10) are satisfied. In solving the problem we used $h_0 = 0{\cdot}1$; $\nu = 0{\cdot}3$; $R = 100$; $N = 15$; $M = 41$. The values of α and β are given in Tables 2 and 3. These Tables contain the values of the deflection and the bending moment, respectively.

From Table 2 it is seen that variation of α in the interval $-0{\cdot}3 \cdots 0{\cdot}3$ leads to an increase in the deflection at fixed values of β. So, at $\beta = 0$ and $x = 0{\cdot}5$; $y = 0{\cdot}5$ when the deflection values are maximal, their ratio is equal to 1:1·23:1·51. The mean value corresponds to a plate with constant thickness. Similar ratios are obtained at $\beta = -0{\cdot}3$ and $\beta = 0{\cdot}3$. One can also see how the deflection varies for decreasing in x and y.

From Table 3 it follows that with the thickness of the plate being increased at the center at $\beta = -0{\cdot}3$, the bending moment also increases, and in turn, with the thickness at the center at $\beta = 0{\cdot}3$ being decreased, the bending moment decreases. It characterizes the variation in a plate stress state at the noted zone. So, the stress-strain state of a plate can be influenced by variation in thickness with its weight being unchanged.

Table 2. Values of the deflection wE/q_0

β	α	$y = 0{\cdot}2$		$y = 0{\cdot}5$	
		$x = 0{\cdot}2$	$x = 0{\cdot}5$	$x = 0{\cdot}2$	$x = 0{\cdot}5$
−0·3	−0·3	4·016	6·138	7·985	12·154
	0·0	4·492	7·641	8·763	14·909
	0·3	5·076	9·603	9·692	18·399
0·0	−0·3	3·152	4·758	7·743	11·582
	0·0	3·473	5·908	8·366	14·233
	0·3	3·831	7·372	9·040	17·540
0·3	−0·3	2·528	3·761	7·789	11·407
	0·0	2·755	4·688	8·321	14·156
	0·3	2·990	5·868	8·856	17·637

Table 3. Values of the bending moment $M_x 10^2/q_0$

β	α	$y = 0{\cdot}2$		$y = 0{\cdot}5$	
		$x = 0{\cdot}2$	$x = 0{\cdot}5$	$x = 0{\cdot}2$	$x = 0{\cdot}5$
−0·3	−0·3	0·598	0·857	1·902	2·741
	0·0	0·529	0·900	1·617	2·754
	0·3	0·383	0·878	1·059	2·570
0·0	−0·3	0·382	0·297	1·346	1·859
	0·0	0·294	0·499	1·117	1·913
	0·3	0·121	0·548	0·664	1·802
0·3	−0·3	0·194	0·294	0·932	1·234
	0·0	0·091	0·154	0·760	1·290
	0·3	−0·102	0·269	0·411	1·261

5. Conclusions

In conclusion, it should be noted that the method proposed in this paper provides a tool for constructing solutions of the two-dimensional boundary-value problem for plates and elastic bodies. These solutions reflect the periodicity with respect to the resolving functions. In practice, the above solutions of the problem may be calculated to sufficient accuracy. Together with the boundary-value problems on bending of variable-thickness rectangular plates, this approach can be extended to the problems of deformation of noncircular cylindrical thin shells and noncircular hollow cylinders in a spatial formulation.

Besides, the method being based on a continuous scheme, makes it possible to obtain the approximate analytical solution of the problem. This problem can not be solved by projective or variational methods. Also the proposed method can be used for solving spatial problems regarding the stress state of noncircular hollow cylinders [13]. Some other approaches to the solution of problems of the theory of plates and shells are presented in [14].

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